Part 2

Part 2 contains:

- Some theory
- A hands on explanation

Please ask questions!

Important Words

- A Monomial is a finite product of unknowns.

$$\mathrm{x}$$
$$\mathrm{xy^5z^3}$$

- A Polynomial is a finite sum of monomials.

$$\mathrm{4x^5 + 3x^3 + 1}$$
$$\mathrm{xyz^5 + 7x^2y + 1}$$

Ideal

The ideal of $\{f_i\}$ is the set

$$I(\{f_i\}) = \left\{ \sum_{\text{finite}} f_j p_j \mid f_j \in \{f_i\}, p_j \in \mathbb{C}[x] \right\}$$

- Contains all polynomials we can generate.
- All the polynomials in the ideal are zero on our zero set.
- Tells us about the number of solutions.
- Contains an infinite number of polynomials.
- If the number of solutions is finite there is a monovariate polynomial in this set.

Polynomial Division and rests

- For univariate division is well defined
- For multivariate division the order of monomials is a central issue.

$$\frac{\mathrm{y}^2+\mathrm{x}}{\mathrm{x}+\mathrm{y}} = 1 + \frac{\mathrm{y}^2-\mathrm{y}}{\mathrm{x}+\mathrm{y}} = \mathrm{y} + \frac{-\mathrm{xy}+\mathrm{x}}{\mathrm{x}+\mathrm{y}}$$

Is $\mathrm{x} > \mathrm{y}^2$?

Monomial Orders

Polynomial division requires that the monomials have an order. There exist several monomial orders

- Lexicographic order (phone-book order), very intuitive, gives long computations. Sort by power of x, then by power of y.
- Graded Lexicographic order. Sort first by total degree and then by lexicographic order. More intuitive and quite good for computing Gröbner bases.
- Graded reversed Lexicographic order, very counterintuitive but it is easier to compute a GB.

Gröbner Basis (GB)

- A basis for the ideal.
- Depends on the monomial order chosen.
- Gives the number of solutions.
- There exist algorithms to compute it.
- Gives ways to compute solutions if the number of solutions is finite.
- All leading terms of polynomials of the ideal are divisible by the leading term of some polynomial in the GB.

S-Polynomials

Definition: Given two polynomials f_1 and f_2 with leading terms $x^{\alpha(1)}$ and $x^{\alpha(2)}$. Let x^β be the least common multiple of the leading terms, then

$$S(f_1, f_2) = \frac{x^\beta}{x^{\alpha(1)}} f_1 - \frac{x^\beta}{x^{\alpha(2)}} f_2$$

is the S-polynomial of f_1 and f_2.

- Multiply both polynomials so that their leading terms cancel out.
- Generating S-polynomials is a key step to building a Gröbner basis.

$$S(x^2 + 1, xy + 3) = y(x^2 + 1) - x(xy + 3) = -3x + y$$

Computing a Gröbner basis

There are many algorithms but the main parts are

- Remove linear dependency.
- Remove multiples of known polynomials.
- Generate new S-polynomials.
- Terminate when nothing new can be generated.
- Must use exact arithmetic.
  - Takes time
  - Takes memory

Matrix Representation of polynomials

$$\begin{aligned} 2x^2 + 4xy + y^2 + 8x + y + 1 \\ 2x^2 + 3xy + 2y^2 + 4x + 4y + 4 \end{aligned}$$

can be represented as

$$\begin{bmatrix} 2 & 4 & 1 & 8 & 1 & 1 \\ 2 & 3 & 2 & 4 & 4 & 4 \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \\ x \\ y \\ 1 \end{bmatrix}$$

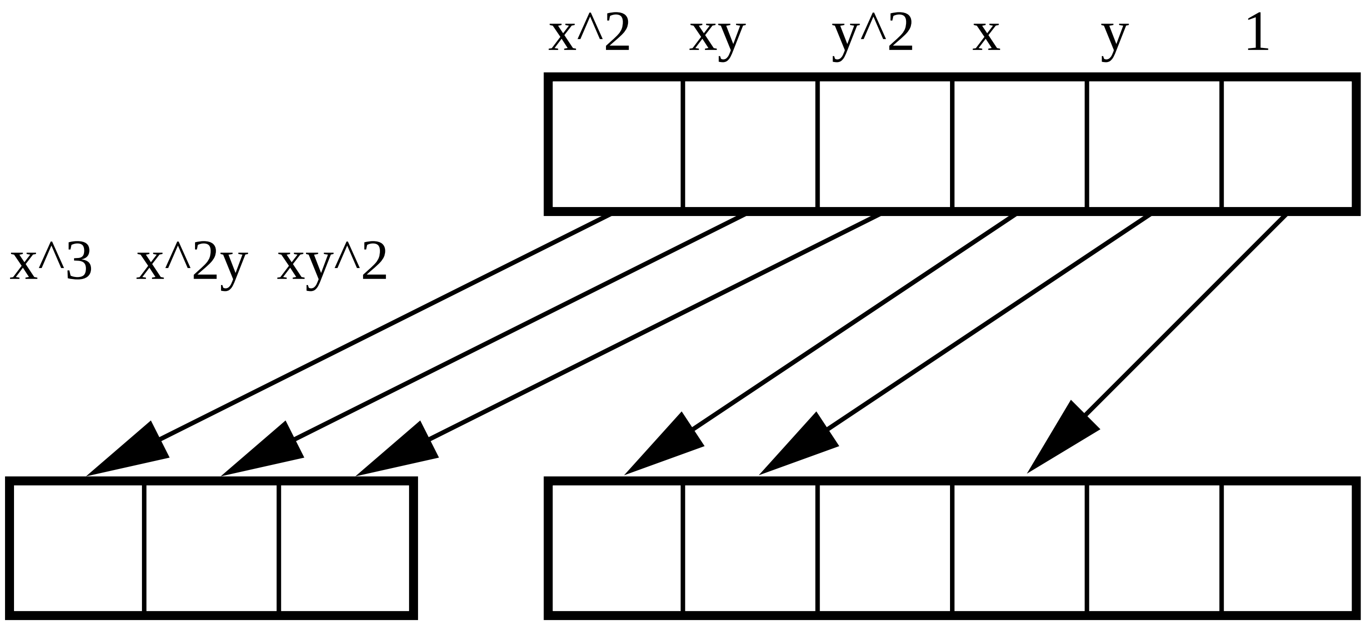
Multiplication with a monomial

- Multiplication is a shift operation

$$(4x + 5y + 6)x = 4x^2 + 5xy + 6x$$

$$\begin{bmatrix} 0 & 0 & 0 & 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \\ x \\ y \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 4 & 5 & 0 & 6 & 0 & 0 \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \\ x \\ y \\ 1 \end{bmatrix}$$

Multiplication with a monomial



- Multiplication is a linear operator.

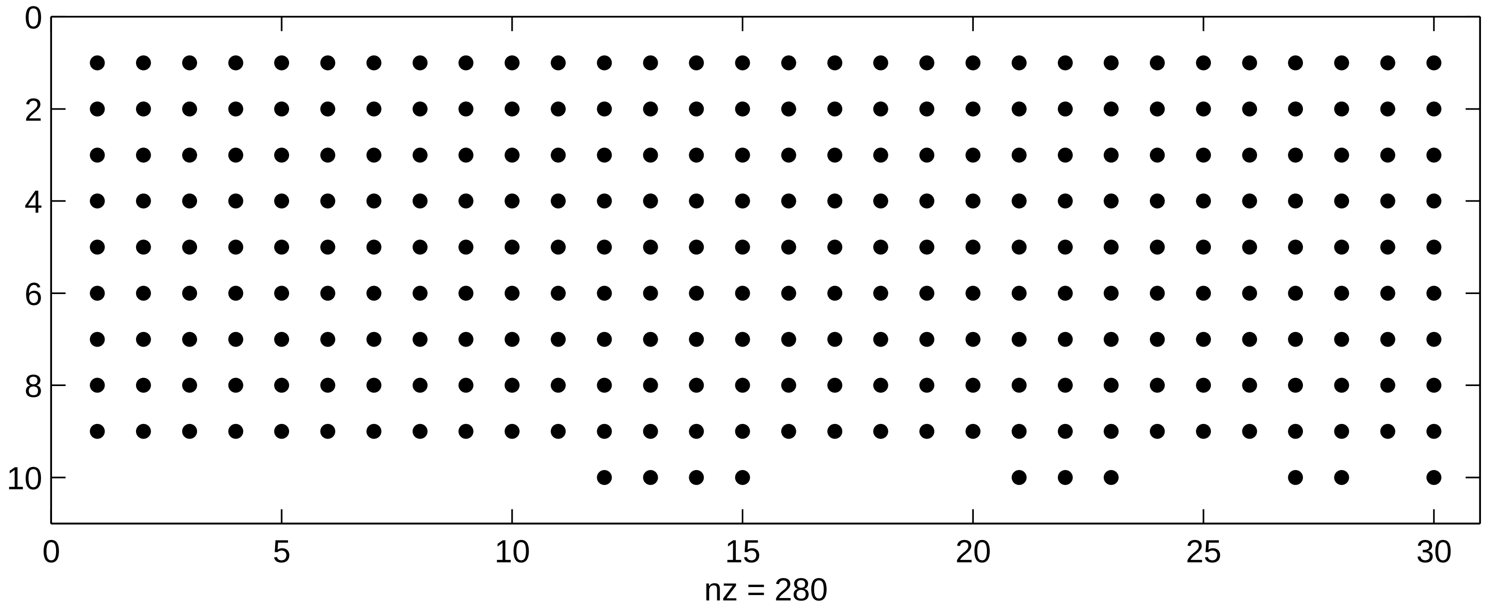
Matrix Based GB Computations

- Add lots of polynomials every time
- Use linear elimination to remove dependencies

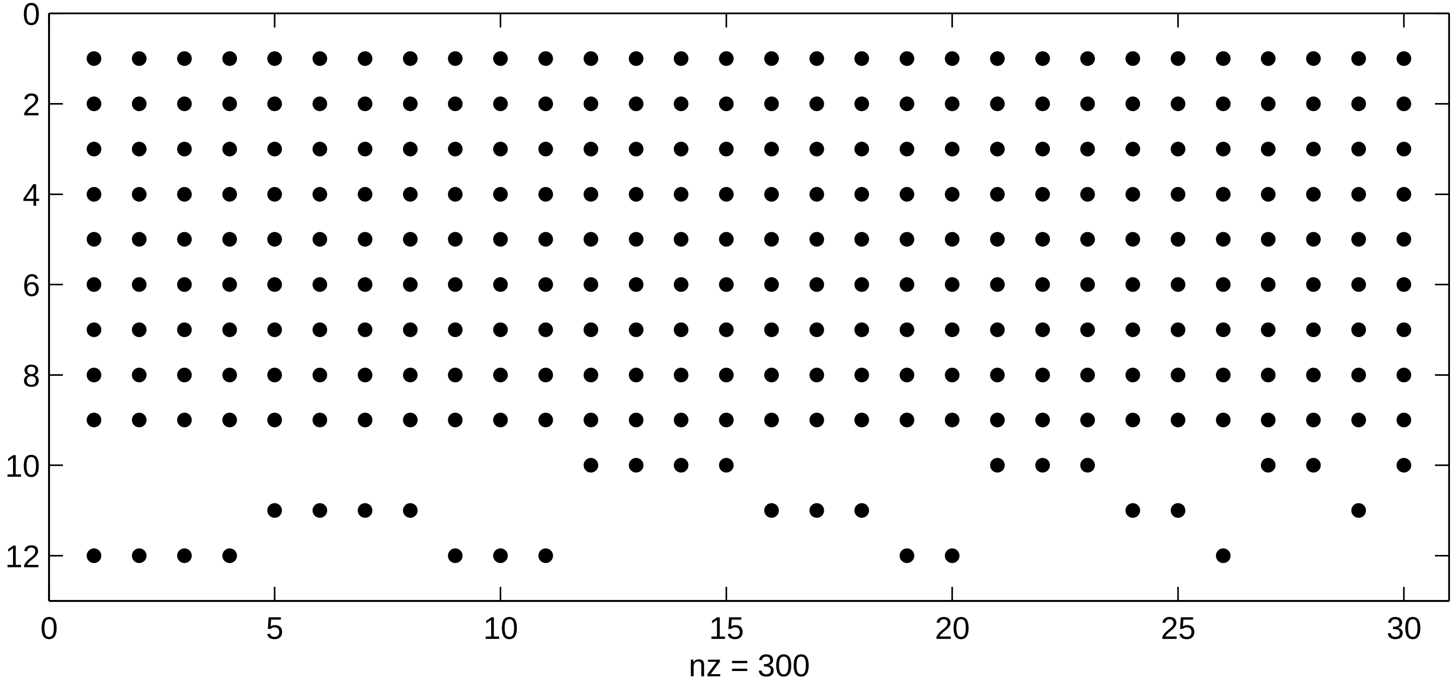
Faugere (F4), Rouillier . . .

- Example: six points and two cameras, unknown focal length.

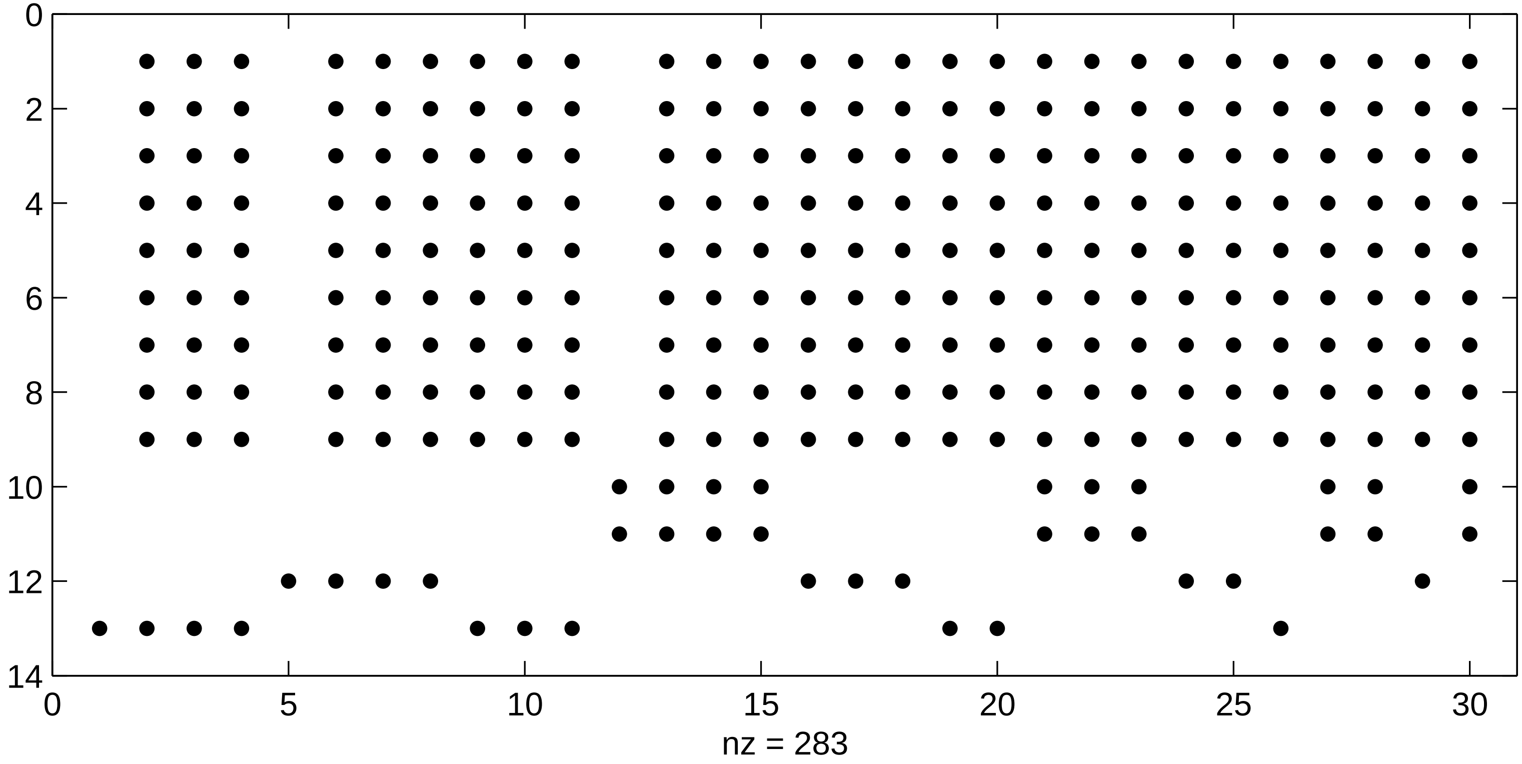
Computational Step



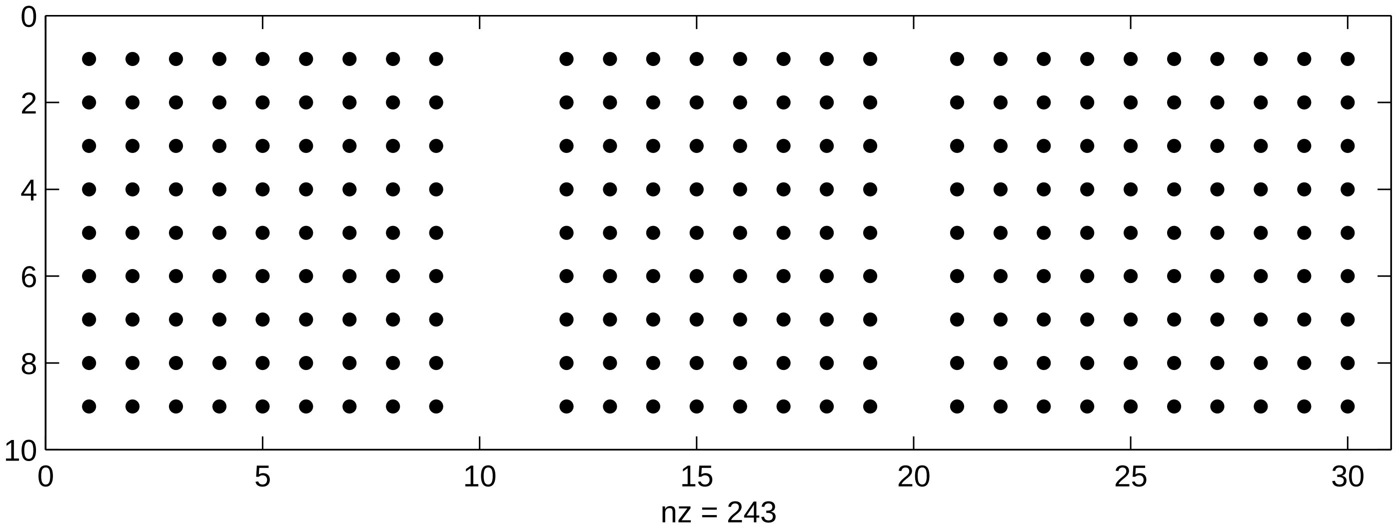
Computational Step



Computational Step



Computational Step



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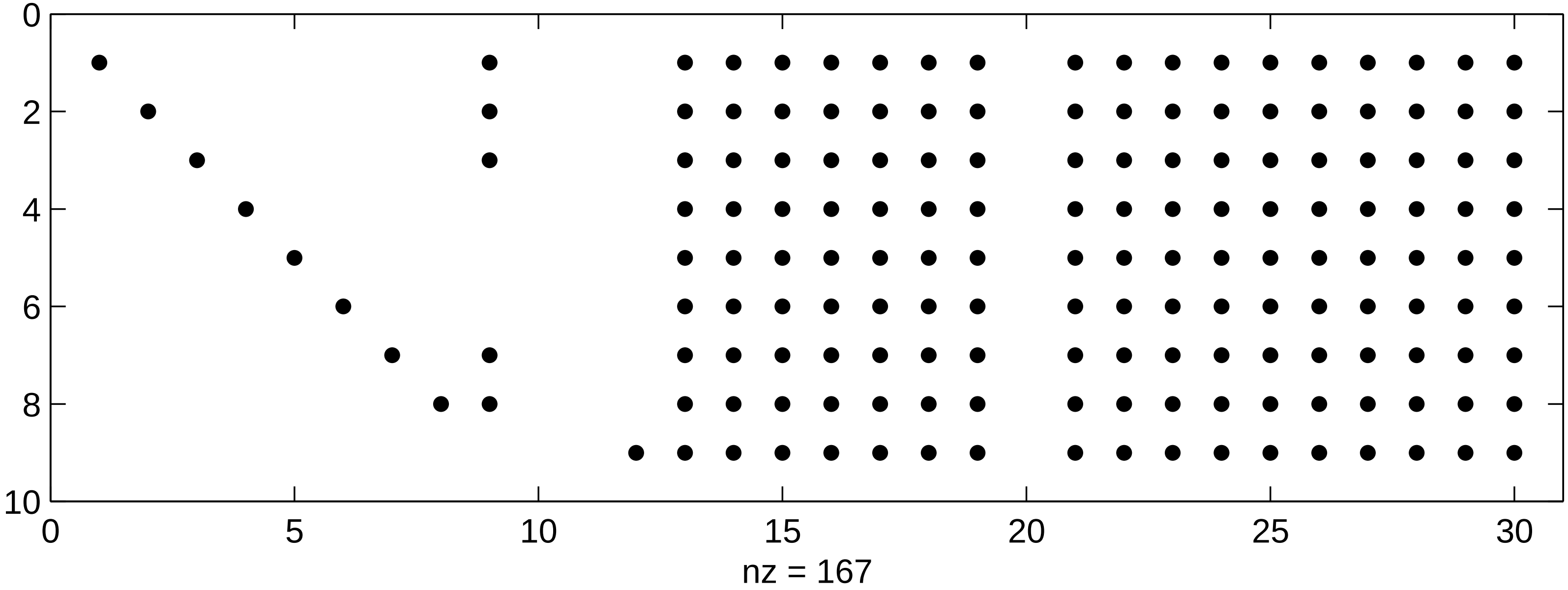
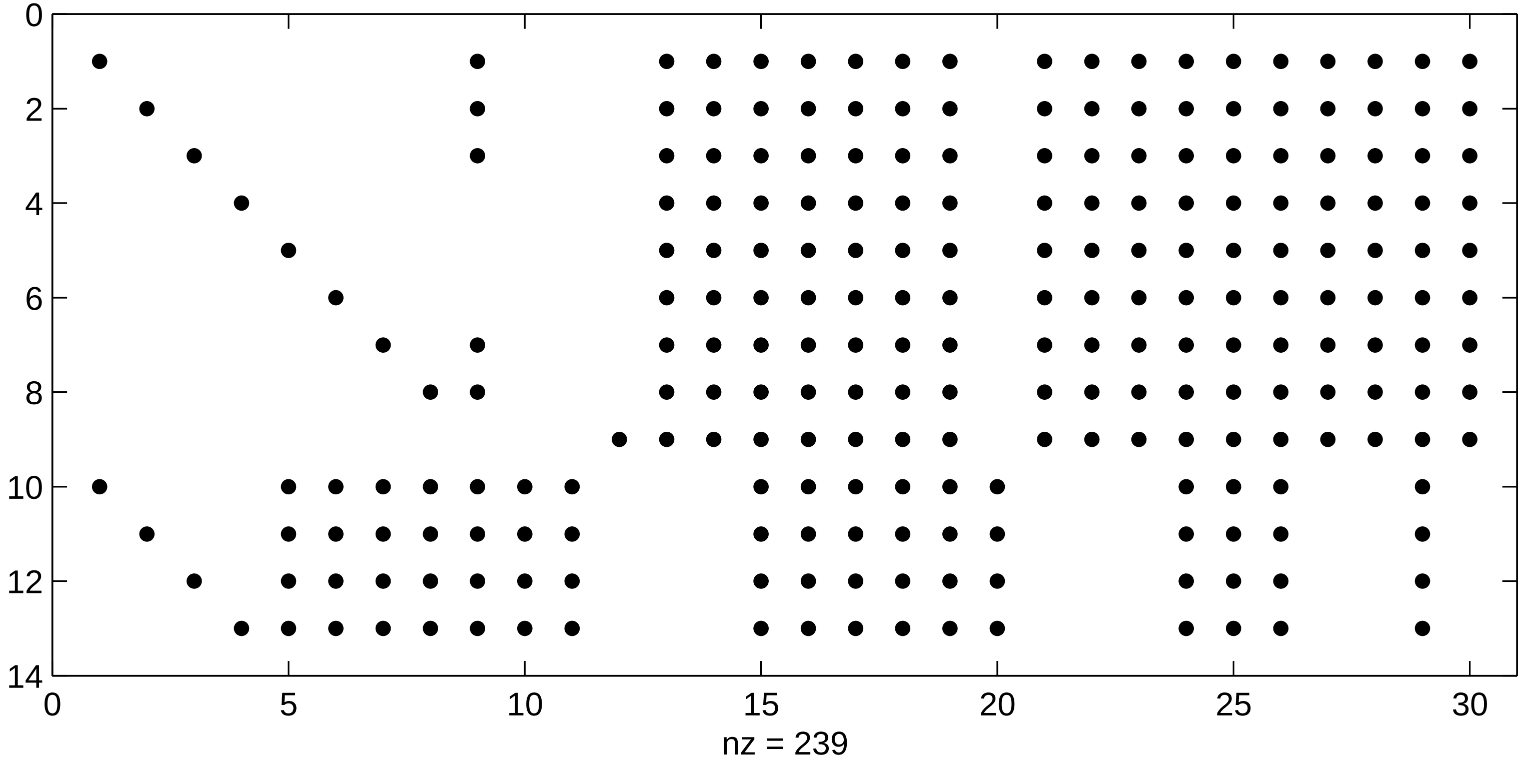


Computational Step



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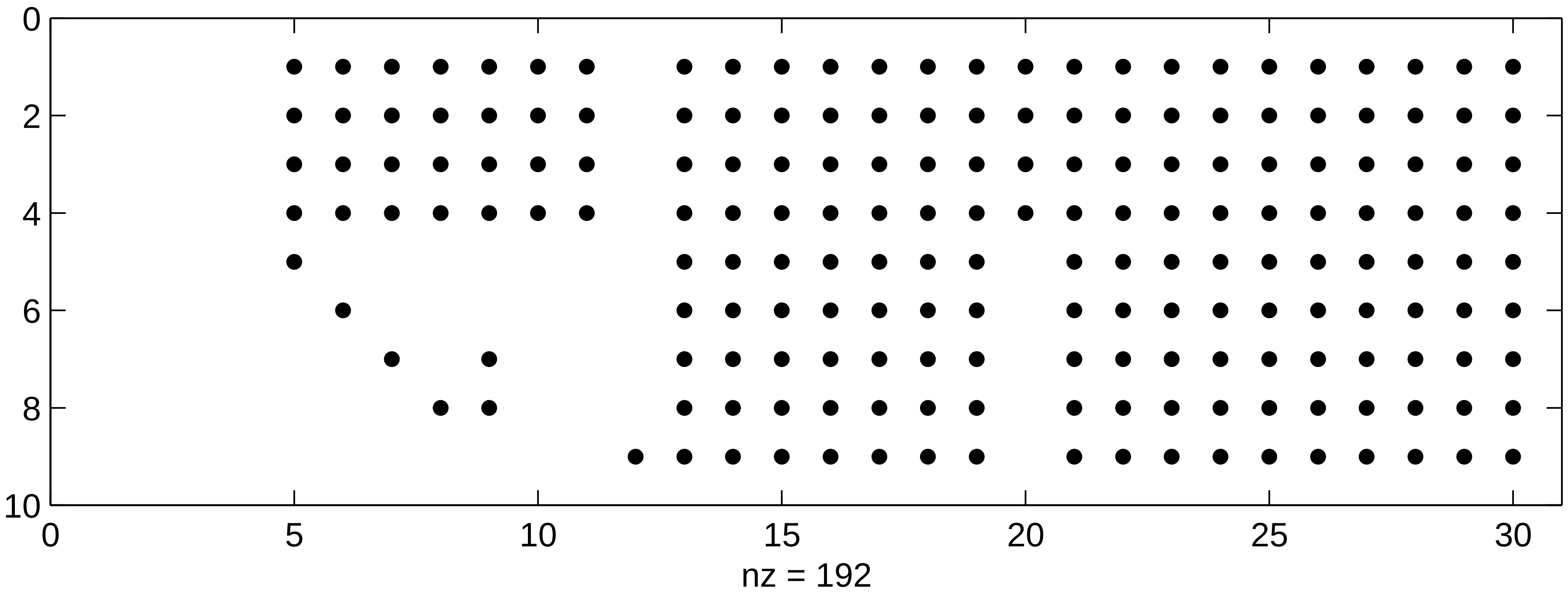
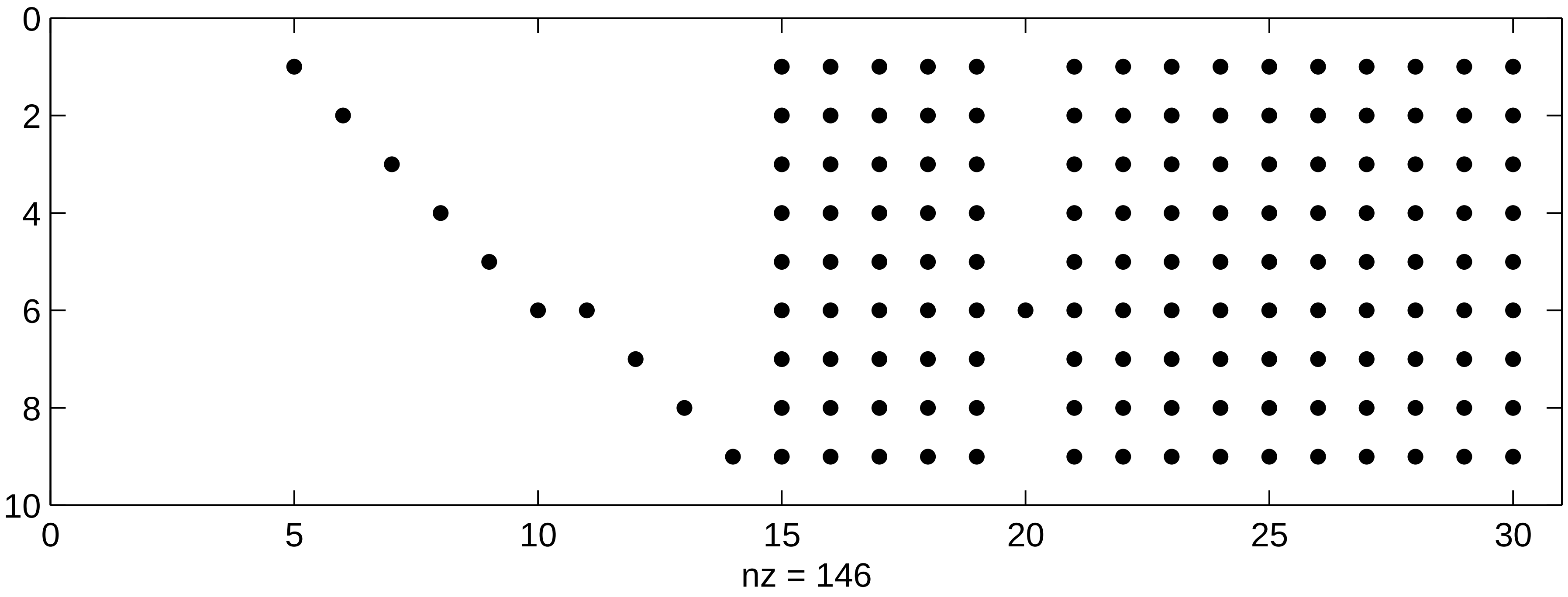
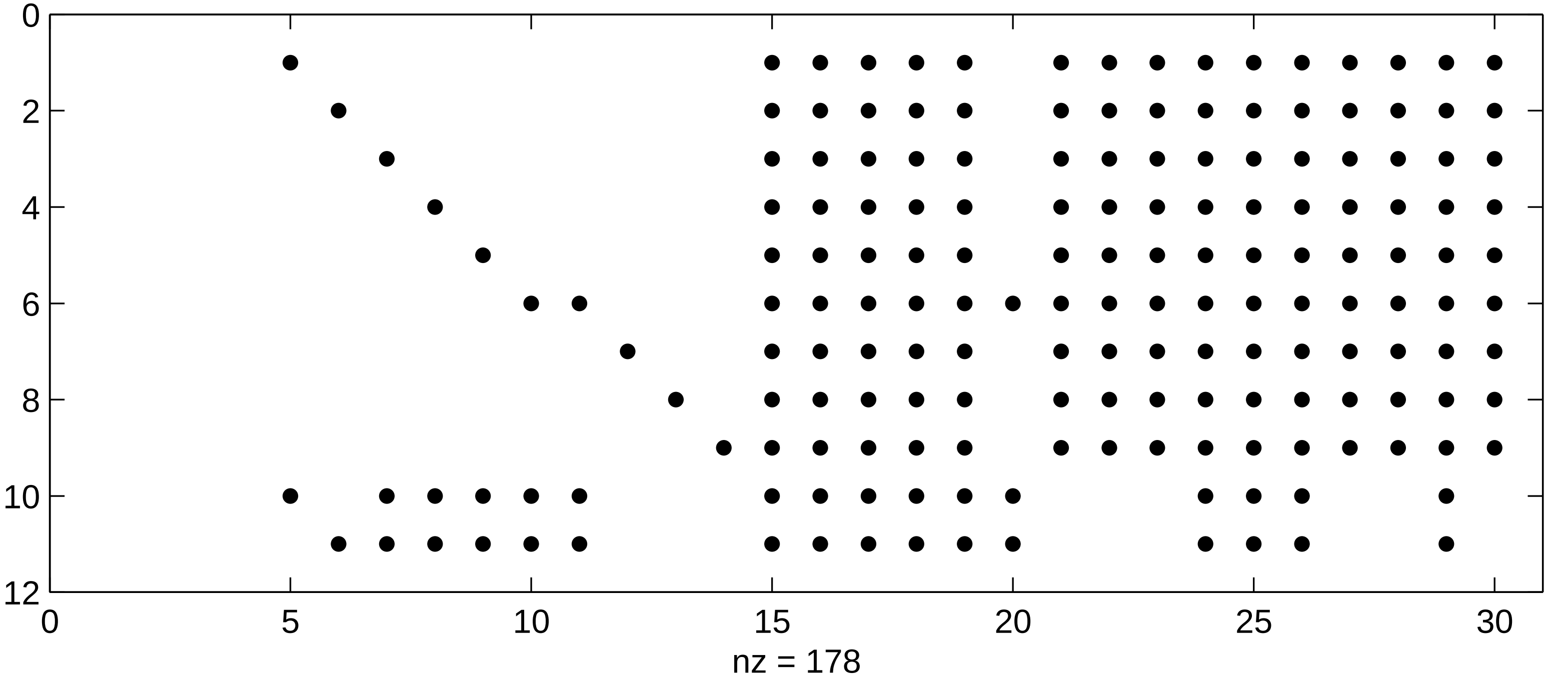


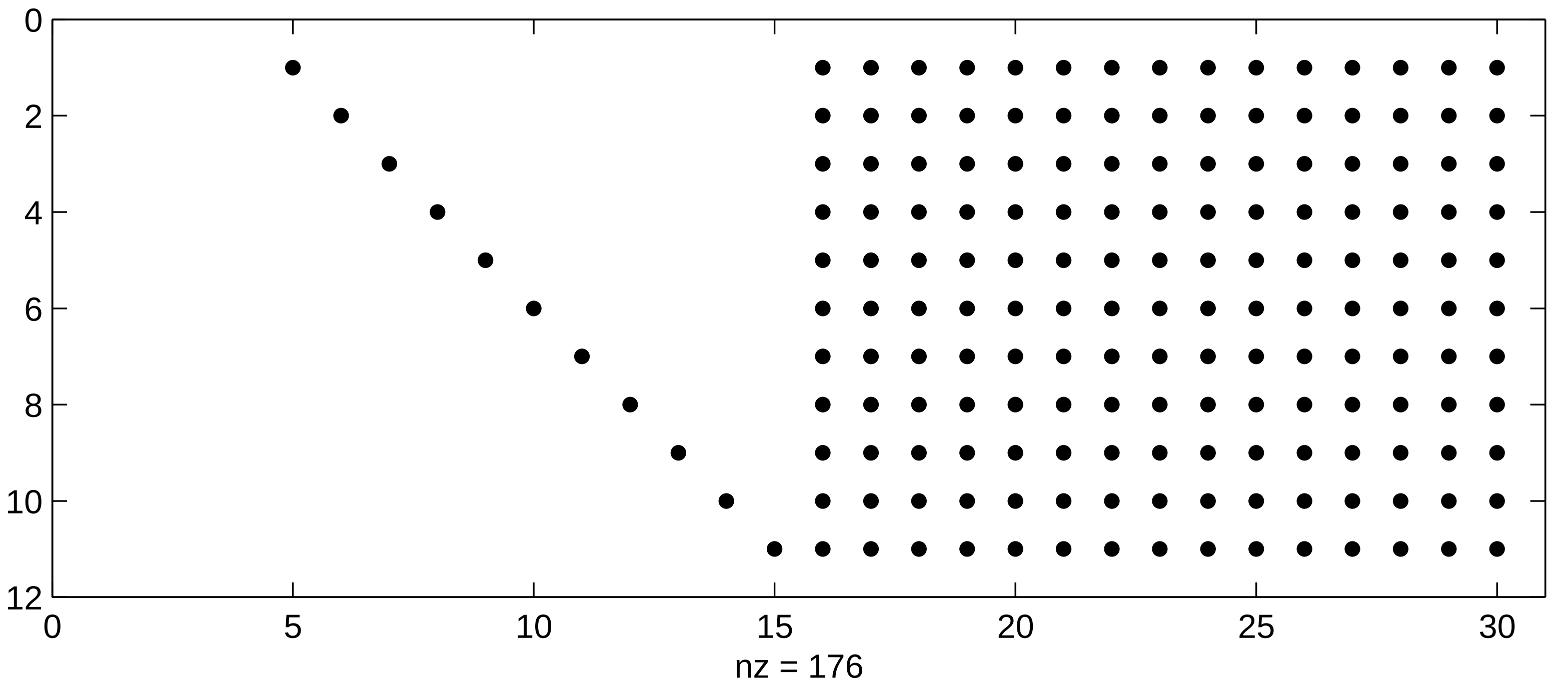
Computational Step



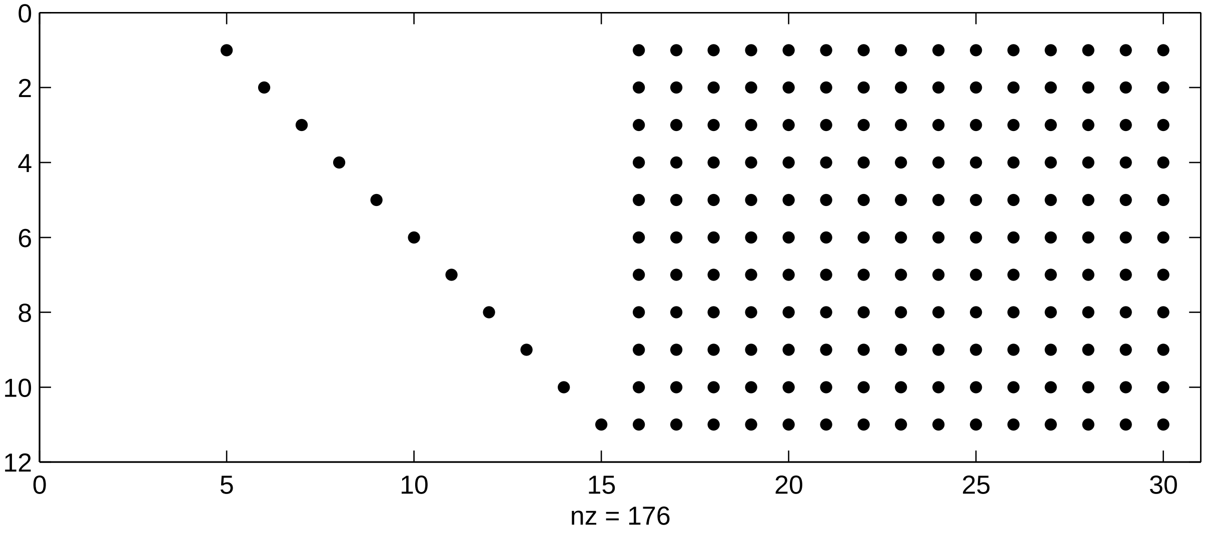
Computational Step

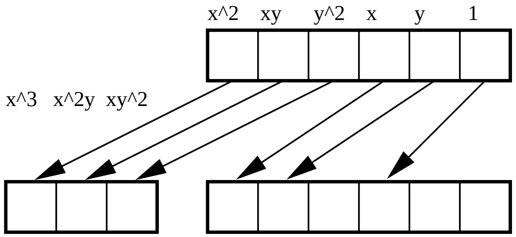


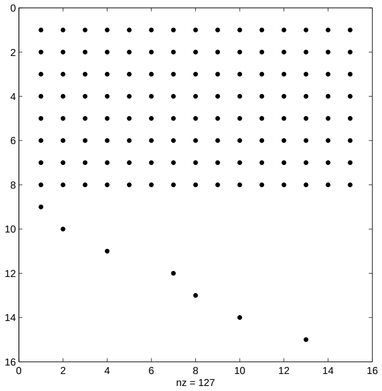
Computational Step



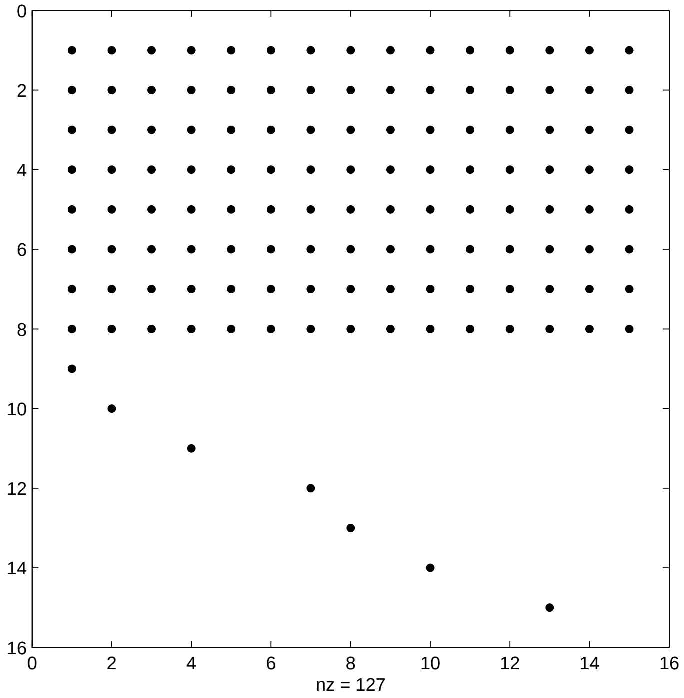
Extracting the Action Matrix







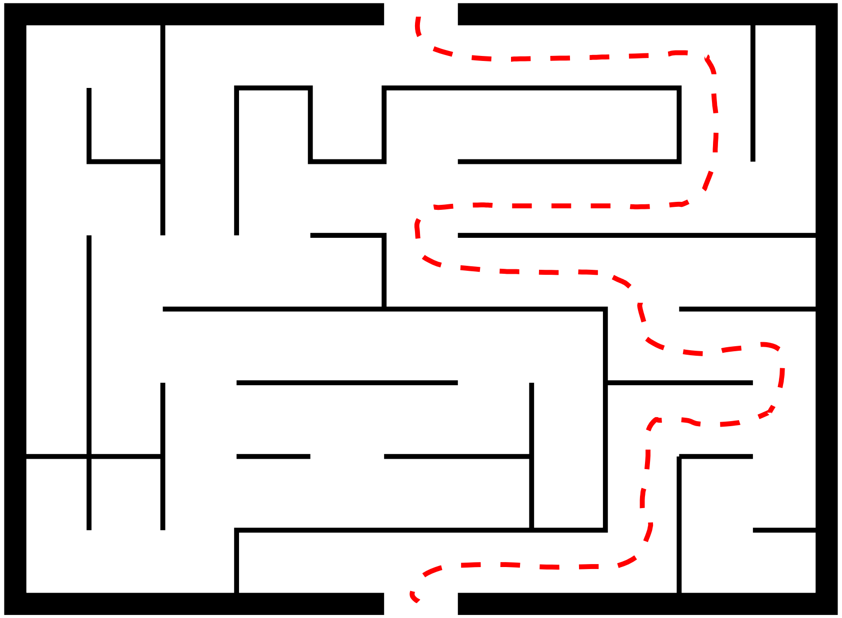
The Action Matrix



- The eigenvectors of this system gives the 15 solutions

The Gröbner Trace

- The trace is always the same in a class of problems.
- Coefficients which become 0 are the walls of a labyrinth.



A finite Field : $\mathbb{Z}_5$

The integers modulo a prime p is a finite field. (Modulo is the rest with division by p.)

- Closed under $+$ and $\cdot$
- Additive and multiplicative inverses

Addition table

	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	5
2	2	3	4	5	6
3	3	4	5	6	0
4	4	5	6	0	1

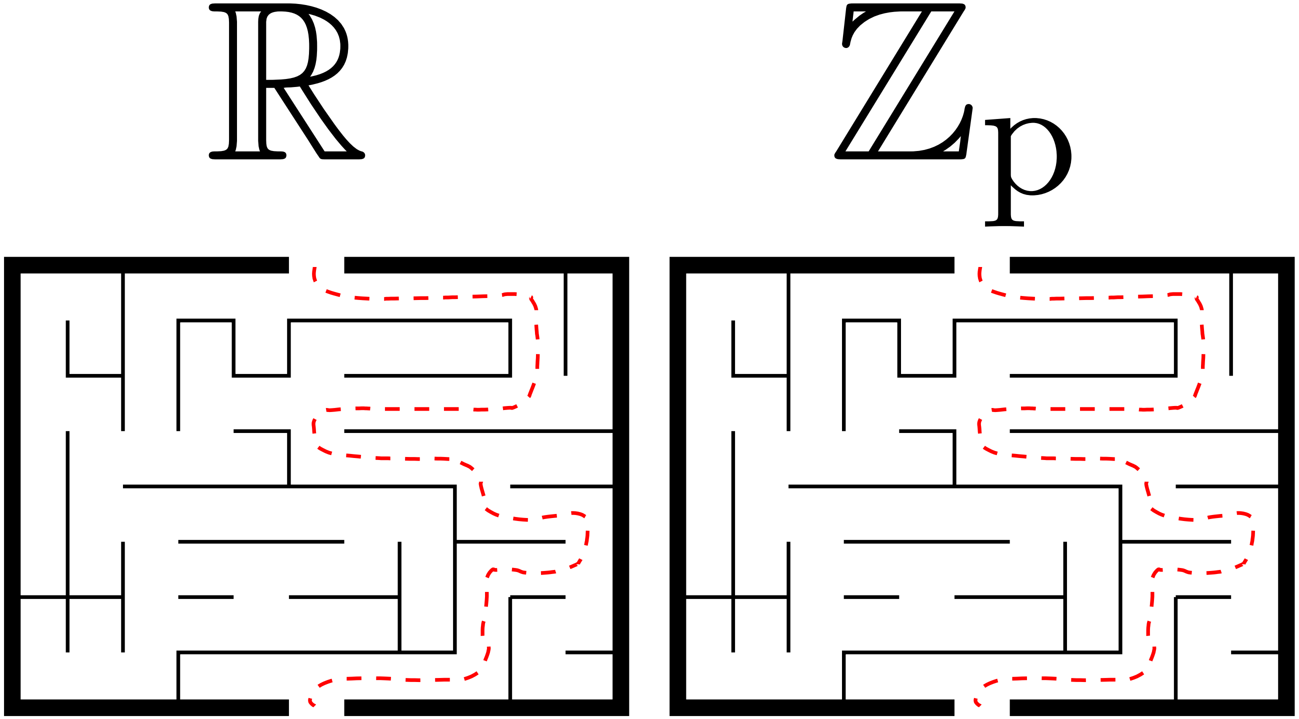
Multiplication table

	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Why use a finite field?

- Exact arithmetic
  - No problem determining if an element is 0
- Constant memory size for all elements of the field
  - It takes more time to run out of memory
- Gives an excuse to brush of some memories of elementary number theory

Fields are usually similar



- $p = 30097$ is a nice prime.
- Using $\mathbb{Z}_p$ makes it easy to compute the trace.

The Quotient Ideal

- $\mathbb{C}[x]/I$, that is the equivalence classes modulo the ideal I.
  1. Take a polynomial in $\mathbb{C}[x]$
  2. Keep computing rests with polynomials in I until this is no longer possible.
  3. The rest is in $\mathbb{C}[x]/I$
- Example $\mathbb{C}[x]/(x^2 - 1)$
  - 2-dimensional space
  - A basis is 1 and x

The action matrix

Consider what happens to the basis polynomials of the quotient ideal $\mathbb{C}[x]/I$ under multiplication by a polynomial f. This is a linear operator A_f.

- Example: Consider $\mathbb{C}[x]/(x^2 - 1)$ with $f(x) = x$. Then $1 \to x$ and $x \to 1$.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- Note the eigenvalues! 1 and -1.

Solution Procedure

1. Pose Problem
2. Count variables and constraints
3. Find GB-trace
4. Compute GB
5. Extract the action matrix
6. Solve Eigenproblem

Helpful tools

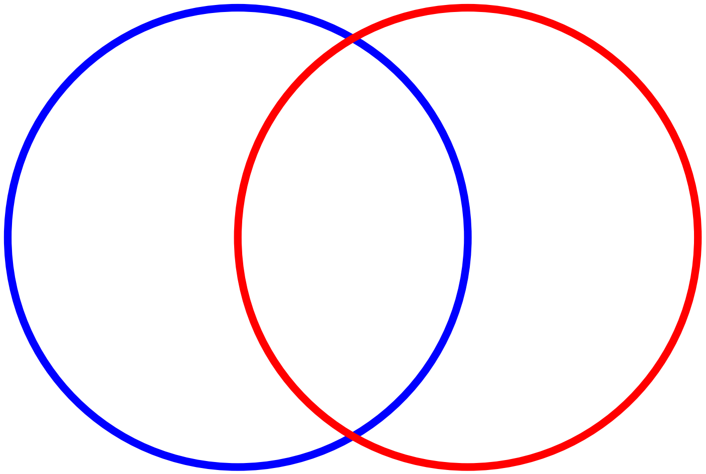
- Macaulay 2,
  `http://www.math.uiuc.edu/Macaulay2/`
  Daniel R. Grayson and Michael E. Stillman, provided under GPL.
- perl
- matlab

Probably all the tools can be substituted for similar tools.

Finding Solutions

- Find all solutions to

$$\begin{cases} x^2 + y^2 & = 1 \\ (x-1)^2 + y^2 & = 1 \end{cases}$$



Howto build an analogous problem 1

```
R = ZZ/30097[x,y];
I = ideal( {x^2+y^2-4 , (x-1)^2+y^2-4 });
dim I , degree I
```

- We get `0,2`

Looking at the trace

```
gbTrace 3
I = ideal( {x^2+y^2-4 , (x-1)^2+y^2-4 });
dim I , degree I
```

Gives

```
mmmr

o12 = (0, 2)
```

m	means an S-polynomial which did not reduce to 0.
r	reduced to 0
o	reduced to 0

Looking at the trace

```
gbTrace 3
I = ideal( {x^2+y^2-4 , (x-1)^2+y^2-4 });
dim I , degree I
```

Gives (filtered a little)

```
--- computing pair (. . x2)  ----
.0  gb 0 = x2<0>+y2<0>-4<0>
--- computing pair (. . x2)  ----
  gb 1 = x<0>+15048<0>
--- computing pair (1 0 x 1 x2)  ----
  gb 2 = y2<0>-7528<0>
--- computing pair (2 1 x y2 xy2 marked)  ----

Number of pairs              = 4
Number of gb elements        = 2
```

Implementing this trace

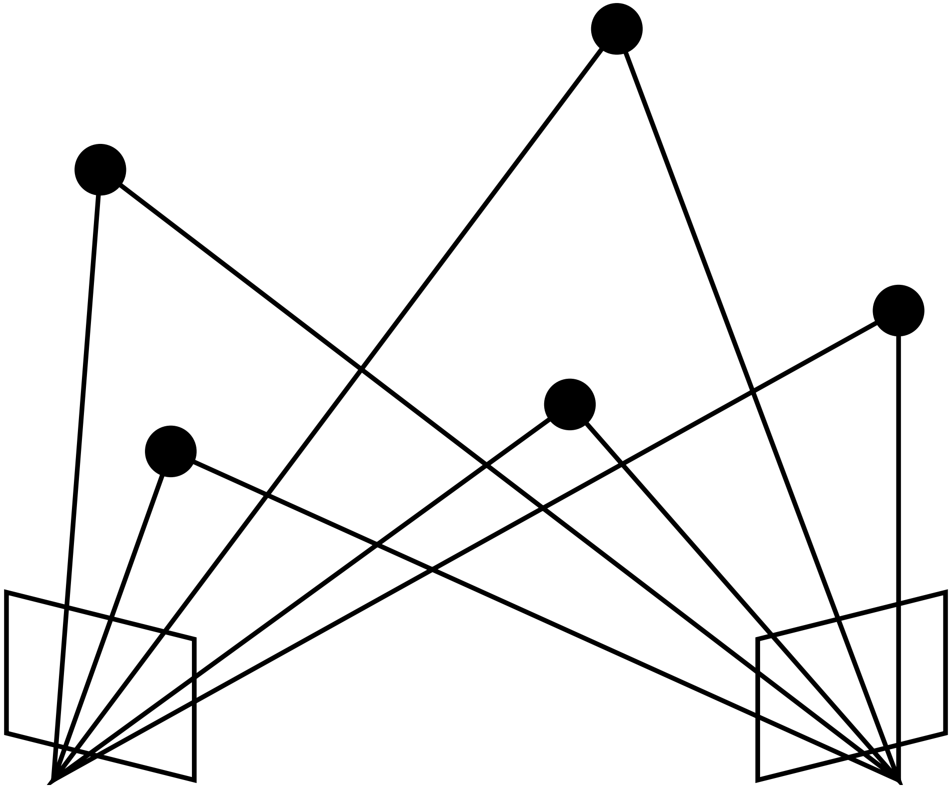
1. Start from $x^2 + y^2 - 1$ and $(x-1)^2 + y^2 - 1$.
2. Linear elimination

$$\begin{bmatrix}1&1&0&-1\\1&1&-2&0\end{bmatrix}\begin{bmatrix}x^2\\y^2\\x\\1\end{bmatrix}\Longrightarrow\begin{bmatrix}1&1&0&-1\\0&0&1&-0.5\end{bmatrix}\begin{bmatrix}x^2\\y^2\\x\\1\end{bmatrix}\Longrightarrow\left\{\begin{array}{rr}x^2+y^2&-1\\x&-0.5\end{array}\right.$$

3. Add $x(x-0.5) = x^2 - 0.5x$ to the basis
4. Linear elimination

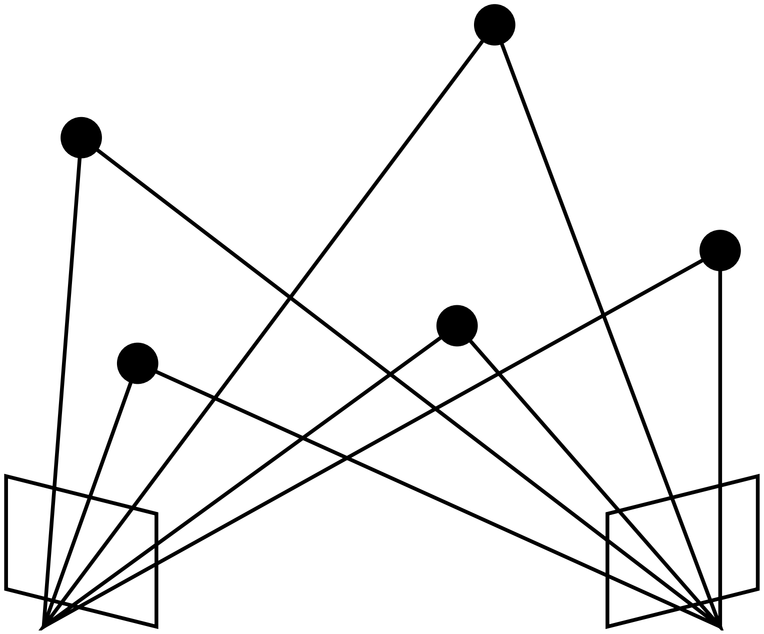
$$\begin{bmatrix}1&1&0&-1\\0&0&1&-0.5\\1&0&-0.5&0\end{bmatrix}\begin{bmatrix}x^2\\y^2\\x\\1\end{bmatrix}\Longrightarrow\begin{bmatrix}1&0&0&0.25\\0&1&0&-0.75\\0&0&1&-0.5\end{bmatrix}\begin{bmatrix}x^2\\y^2\\x\\1\end{bmatrix}\Longrightarrow\left\{\begin{array}{lrrr}x^2&&&+0.25\\&y^2&&-0.75\\&&x&-0.5\end{array}\right.$$

Relative Pose



- Observation Equation $\lambda_{ij} u_{ij} = P_i U_j$.
- Alternative Observation equation $x^\top F x' = 0$.
- Restrictions on P or F.

Finding Solutions



- Five points and two cameras relative pose

$$\begin{cases} \mathrm{E} = \mathrm{E}_0 + \mathrm{x}\mathrm{E}_1 + \mathrm{y}\mathrm{E}_2 + \mathrm{z}\mathrm{E}_3 \\ 2\mathrm{E}\mathrm{E}^\top\mathrm{E} - \mathrm{trace}(\mathrm{E}\mathrm{E}^t)\mathrm{E} = 0 \\ \det(\mathrm{E}) = 0 \end{cases}$$

Howto build an analogous problem 2

Investigating the analogous problem for the 5-point problem.

```
R=ZZ/30097[x_0..x_2];
t={1_R,x_0,x_1,x_2};
E=fold((a,b)->a+b,apply(4,i->t_i*random(R^3,R^3)));
EEt = E*transpose(E);
I = ideal( 2*EEt*E-trace(EEt)*E )+ ideal det E;
gbTrace 3
dim I , degree I
```

Howto build an analogous problem

- Set up the finite field (3 variables)

```
R = ZZ/30097[x_0..x_2];
```

- Declare a list

```
t = {1_R,x_0,x_1,x_2};
```

- Set $E = E_3 + \sum_{i=0}^{2} t_i E_3$ where E_i random.

```
E=fold((a,b)->a+b,apply(4,i->t_i*random(R^3,R^3))
```

- Write down the equations, take the ideal

```
EEt = E*transpose(E);
I = ideal( 2*EEt*E-trace(EEt)*E )+ ideal det E;
```

- Set the amount of information from the computation

```
gbTrace 3
```

- Compute the dimension and degree of the ideal

```
dim I , degree I
```

The output

```
i7 : dim I , degree I
mmmmmmmmmmooooooooooooooo

o7 = (0, 10)
```

There are 10 solutions of dimension 0

More output

Set `gbTrace 100`, rerun computations and filter

```
--- computing pair (. . x_0^3)  ----
.0  gb 0 = x_0^3
     .
     .
--- computing pair (. . x_0^3)  ----
  gb 9 = x_2^3
--- computing pair (9 8 x_1 x_2 x_1x_2^3)  ----
     .
     .
.21--- computing pair (1 0 x_0 x_1 x_0^3x_1)  ----
Number of gb elements       = 10
Number of pairs computed    = 25
```

We see that no new S-polynomials where generated.

Implementing this trace

1. Set the 10 equations into a system with coefficients ordered in GrLex order.

```
MONS = [3 2 2 1 1 1 0 0 0 0 2 1 1 0 0 0 1 0 0 0;
        0 1 0 2 1 0 3 2 1 0 0 1 0 2 1 0 0 1 0 0;
        0 0 1 0 1 2 0 1 2 3 0 0 1 0 1 2 0 0 1 0];
```

2. Perform Gaussian elimination

$$\begin{bmatrix} \mathrm{I} & \mathrm{M} \end{bmatrix}$$

3. We now have a Gröbner basis

To get a complete solver we have to extract the action matrix.

`http://www.vis.uky.edu/~stewe/FIVEPOINT/`

The filter I used

```
rm tt
cat class25_new.m2 | M2 > tt
cat tt|perl -p -e 's/gb/##gb/;s/comput/##comput/;s/\<0\>.*//'
      |egrep "##" | perl -p -e 's/##//'  | egrep -v marked
```

(one line-break inserted where there should be none)

Conclusions from M2

- Study the number of solutions
- Study if the trace looks possible
- This is an iterative process

The Problem Formulation is Very Important

- Get as many equations as possible
- Keep the degree as low as possible
- Keep the number of variables as low as possible

Tricks

- Use fundamental/essential matrix if possible.
- Use determinants to remove variables.
- ...

6 point, unknown f

- Stewénius, Nistér, Kahl and Shaffalitzky, CVPR2005.
- Using the trace constraint and the rank constraint building a GB is very straight forward.
- 15 solutions

6 point, unknown f

$$F = F0 + l_1F_1 + l_2F_2$$

```
KK = ZZ / 30097;
R = KK[l_1,l_2,p];
F0 = random(R^3,R^3);
F1 = random(R^3,R^3);
F2 = random(R^3,R^3);
F = F0 + l_1*F1 + l_2*F2;
```

6 point, unknown f

$$E = K^\top F K$$

```
P = matrix {{1,0,0},{0,1,0},{0,0,p}};
tr = X -> det( X_{0}^{0} +X_{1}^{1}+X_{2}^{2});
EQ1=2*F*P*transpose(F)*P*F-tr(P*F*P*transpose(F))*F;
EQ2=det( F);
I = ideal EQ1 + ideal EQ2;
gbTrace 3;
dim I , degree I
```

Do not drop equations!

- Variable focal length (10 equations)

```
mmmmmmmmmmmmmmmmmmmmmmmmooooooooo
oooooooorooooroorrrrrororoor
15 solutions
```

- Variable focal length (2 equations)

```
mmmmmmmmmmmmmmmmommmmmmooooooooo
oooorooooroorrrrr
27 solutions
```

Generalized Camera in M2

- Generalized Camera (good problem formulation)

```
mmmmmmmmmmooomoomommooooooooommmmmmmm
mmmmmmommoommmooooooooooooooooooomooo
ooooooooooororrrororoo
```

64 solutions

- Generalized Camera (bad problem formulation)

```
mmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmm
mmmmmmmmmmmommmommmmommmmmmmoommmmmmm
mmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmmm
            ...... (total 168 lines)
rrrrrorrrrrrrrorrrrrrrrrrrrorrrrrrrrrr
```

64 solutions

Implementing a Trace

We have a formulation where the trace looks ok

1. Write a matrix version in `M2`.
2. Build the equations in real numbers
3. Port the matrix version to `Matlab` or `C`

A Matrix Version in M2

- M2 was not meant for matrix operations
- Need to hack our own Gaussian elimination
  - It can be done using the Gröbner engine
  - It is also possible to hack it using special commands. Please email me for source.

Extracting the equations

Two alternatives

- Symbolically expand the equations and identify the coefficients (slower)
- Build a vector based multiplication, much like convolution. (faster)

It is strongly recommended to use some symbolic toolbox for this. For more complex expressions the expansion can be very long!

Porting a solver

1. Put the original equations into a matrix
2. Derive the multiplication matrices
3. Work carefully. check against a problem where the ground truth is known.

Setting up the monomials in GrevLex order

```
function MONS = build_MONS_3var(n)
MONS =[];
for i1=0:n
   for i2=0:n
      if i1+i2 <= n
         MONS = [MONS [i1;i2]];
      end
   end
end
[dummy,I]=sortrows(  [sum(MONS);  -MONS(end:-1:1,:) ]
MONS = MONS(:,I(end:-1:1));
```

Build a multiplication matrix

```
function M = build_mult_matrix( MONS_from,MONS_to,mul
% MONS_* matrices where the columns represent monomia
% mult is a column matrix representing a monomial
%
mult = mult(:)';
M =[];
for i=1:size(MONS_from,2)
   p = MONS_from(:,i)';
   M=[M ismember(  MONS_to', p+mult , 'rows' ) ];
end
M = M';
```

Extract the Action Matrix

For the 5-point the action matrix is read by the following commands.

```
M = -A([1 2 3 5 6 8], :);
M(7,1) = 1;
M(8,2) = 1;
M(9,4) = 1;
M(10,7) = 1;
```

- Use code to build this code! (does not fit this page)

Action Matrix to Solutions

```
[V,D] = eig(M );
SOLS =   V(7:9,:)./(ones(3,1)*V(10,:));
```

and then back-substitute

```
Evec = EE*[SOLS ; ones(1,10 ) ];
```

planarly restricted generalized camera

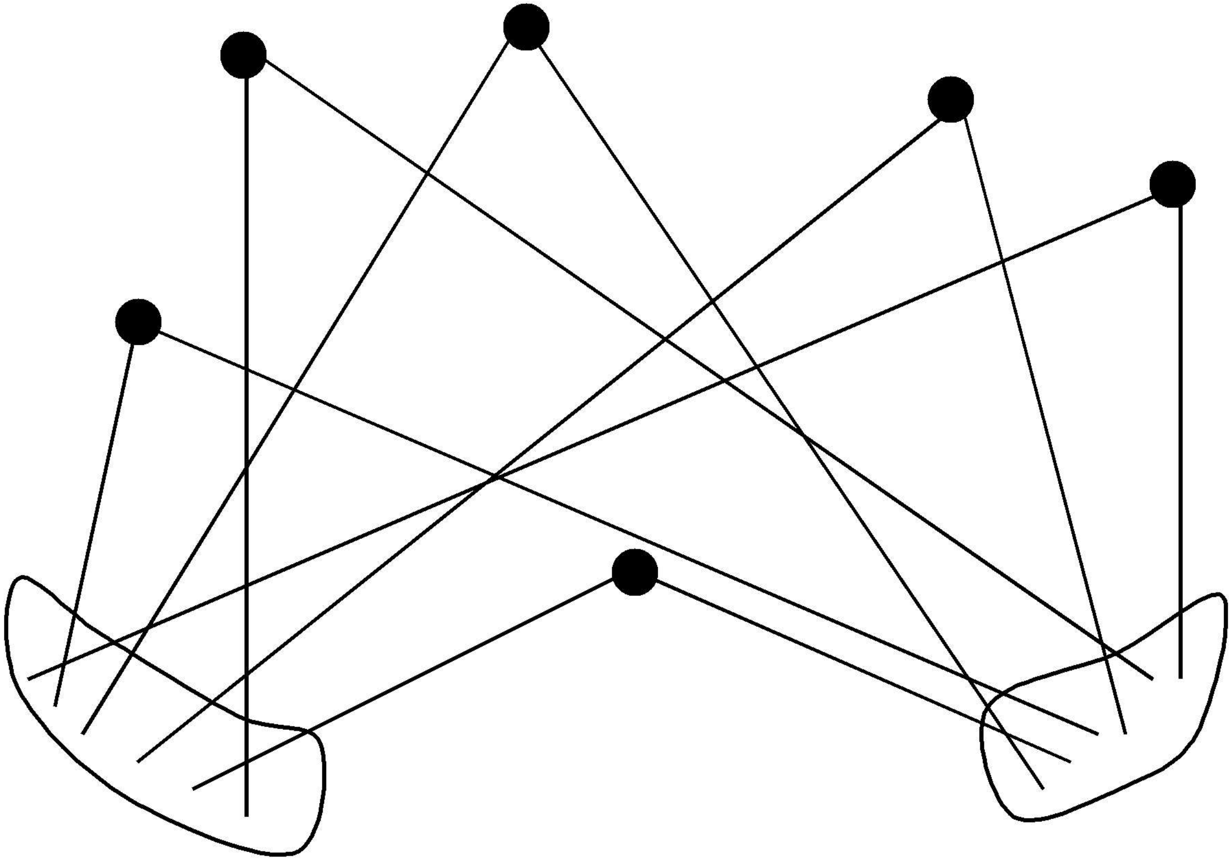
- H. Stewénius and K. Åstrom ECCV2004, before we fully understood how to work with GB.
- There are two easy minimal relative pose problems.
- This case is suitable as an easy exercise to begin with.

Relative Pose generalized camera

- There are 2 minimal cases, 64 solutions and 1320 solutions.
- For the easy problem:
  - Stewénius, Nistér, Oskarsson and Åström at OMNIVIS 2005.
  - Departing from the generalized epipolar constraint.
  - Requires a rather hard-to-find rewrite

The Generalized Camera

- A Generalized Camera maps from points in space to lines in the camera coordinate system.
  $(\mathrm{X}, \mathrm{T}) \rightarrow (\mathrm{q}, \mathrm{q}')$



The Generalized Epipolar Constraint

- $R_1 = I$
- R_2 is the rotation at time i
- t_i is the translation at time i
- (q_i, q_i') is the observation in the camera

$$q_2^\top R_2 q_1' + q_2^\top (R_2 [t_1]_\times - [t_2]_\times R_2) q_1 + q_1^\top R_2^\top q_2' = 0$$

- This gives a polynomial

Parameterizing with depth i

Let $\mathrm{U}_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$ define the coordinate system. Then

$$\mathrm{t}_\mathrm{j} = \mathrm{q}'_{\mathrm{kj}} \times \mathrm{q}_{\mathrm{kj}} + \lambda_{\mathrm{kj}} \mathrm{q}_{\mathrm{kj}} \quad \mathrm{j} = 1, 2, \qquad \underbrace{\mathrm{F}^\mathrm{k}(\mathrm{v})}_{5\times 3} \begin{bmatrix} \lambda_{\mathrm{k}1} \\ \lambda_{\mathrm{k}2} \\ 1 \end{bmatrix} = 0$$

- $\mathrm{F}^\mathrm{k}(\mathrm{v})$ is quadratic in v
- rank condition gives 6th order equations on v

$$\begin{bmatrix} [2] & [2] & [2] \\ [2] & [2] & [2] \\ [2] & [2] & [2] \\ [2] & [2] & [2] \\ [2] & [2] & [2] \end{bmatrix}$$

Using a second asymmetry

- Choosing $U_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$ gives 10 linearly independent equations.
- Choosing $U_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$ gives 10 linearly independent equations.

Together this gives 14 equations

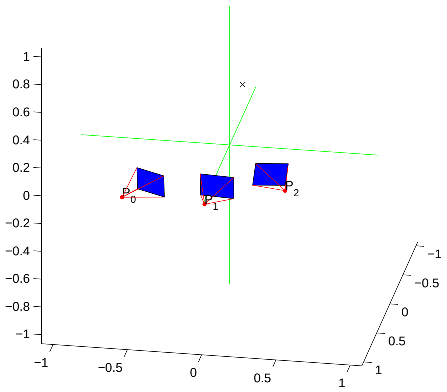
Constructing a GB

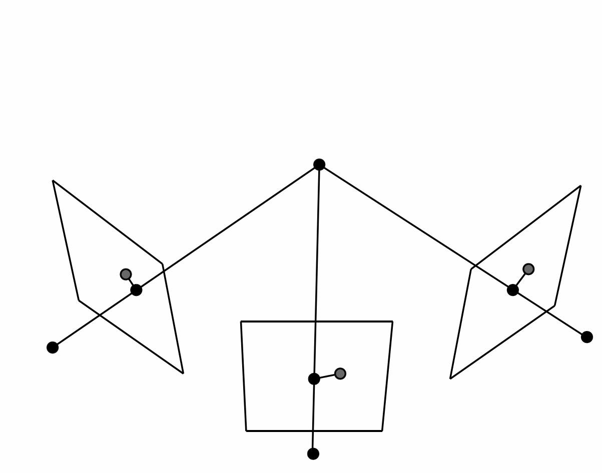
1. Add the polynomials $\mathrm{v_j f_i}$ there are now 56 polynomials
2. Compute the missing polynomial as an S-polynomial

We know have a GrevLex Gröbner-basis.

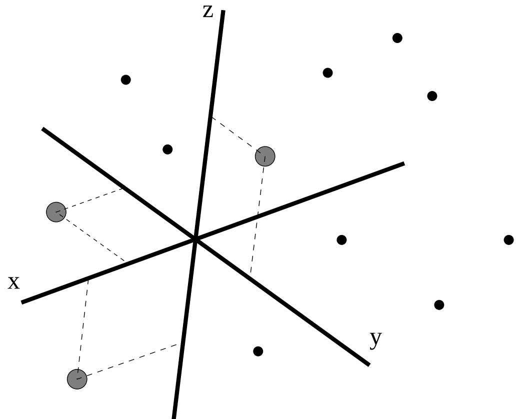
- As there are 64 inner points to the Gröbner basis the action matrix is 64 × 64. This gives a total of 64 solutions.

3-view triangulation, L_2-norm





- There are 47 stationary points to the target function
- Have to remove unwanted zeros (saturation)



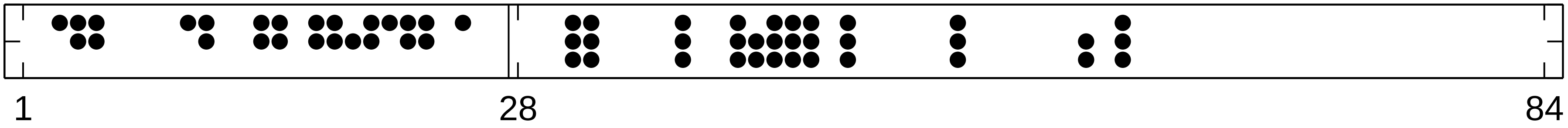
Removing Zeros

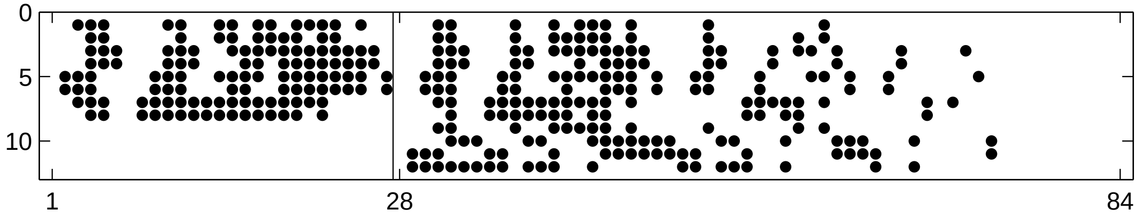
We want to remove the set defined by $f(x) = 0$ from the solutions. Two different possibilities:

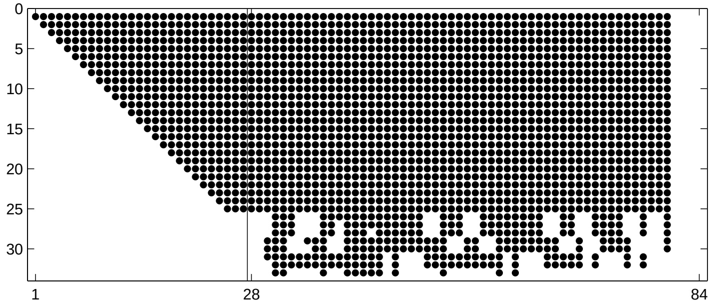
- Rabinovichs trick. Add the synthetic variable s and the polynomial $1 - sf(x)$.
- Divide out the unwanted solutions.
    - Use special monomial order. (Elimination order)

Saturation

- Saturation, three rounds







- Data flow

I_0 Partial Saturation I_1 Partial Saturation I_2 Partial Saturation I_3

Some help

- Construct test to verify results halfway …
- Do not write by hand what the computer can write for you
- Feel free to browse in my poorly commented code archive
- The slides will be available soon

  `http://www.vis.uky.edu/~stewe/`

End

The slides and some code will be available on:

`http://www.vis.uky.edu/~dnister/`

`http://www.vis.uky.edu/~stewe/`